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Timoshenko beam with tuned mass dampers and its design curves

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Abstract

The structural analysis and design of a Timoshenko beam with tuned mass dampers (or TMDs) under a harmonic excitation are presented. The dynamic-stiffness matrix of a Timoshenko-beam element is employed to study the dynamic responses for the whole frequency range. A proposed simplified two-degree-of-freedom system and Hartog's method are employed to study the dynamic characteristics of TMDs. Some important formulas for the design parameters (such as the mass ratio and stiffness ratio at the tuned frequency, the optimal damping ratio of TMDs, and their upper limits) and a series of the design charts are presented for practical applications, if the beam-own damping is absent and all TMDs are identical.

1. Introduction

One of the most simple and economic ways to control the vibration of a beam structure is the so-called tuned mass damper (or TMD) [1–16], which is a single mass attached to the beam structure by viscoelastic material or other mechanism of similar effect. The tuned mass damper can be modelled as a simple mass–spring–dashpot system, the dynamic behaviors of a beam structure with TMDs can then be easily formulated and analyzed by the finite-element method. The dynamic stiffness matrix of an axial-loaded damped Timoshenko beam on viscoelastic foundation has been established by the first author of this paper and applied to many engineering problems [17–27]. All the effects of the rotary inertia of mass, the shear distortion, the constant axial force, the viscoelastic foundation, and the various damping components can be taken into

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account in analysis. By the direct stiffness method as used in the standard structural analysis, the stiffness matrix of an entire beam system can be obtained by the assembly of all the element dynamic-stiffness matrices or partially with the element static-stiffness matrices. The conventional finite element analysis employs the interpolation functions for beam deflection which are static in nature, they are not reliable to define the dynamic behaviors of beam vibration in the high-frequency range. The formulation of the dynamic stiffness matrix of a Timoshenko beam is based on the exact dynamic displacement functions; therefore it is valid for the whole frequency range including the high-frequency (or the high-mode) as well as the low-frequency (or the low-mode) vibrations.

There are three important parameters of a tuned mass damper in design to control the beam vibration, namely the mass ratio, stiffness, and damping ratio. In practice, the mass ratio of the tuned mass damper to the beam is not much greater than 10%; therefore there are only two parameters, i.e., the stiffness and the damping ratio of a tuned mass damper, need to be decided in advance. In order to achieve the best performance of the tuned mass damper, these two parameters should be chosen appropriately to make the beam vibration as small as possible. There are two important methods to determine these two parameters to achieve the best performance of a tuned mass damper, one is presented by Hartog [1] and the other by Jacquot and Hoppe [4]. Both methods can give the values of these two parameters at the optimal condition called the tuned frequency (or the tuned frequency ratio) and the optimal damping ratio of a tuned mass damper, respectively. The optimal condition of Hartog's method is based on that the amplitude of the steady state dynamic response under the harmonic excitation is minimum, whereas the optimal condition of Jacquot and Hoppe's method is based on that the mean-square response under the white-noise random excitation is minimum. Some other methods have also been proposed and the comparison between them can be found in Ref. [16].

In this paper both the dynamic stiffness of a Timoshenko beam and Hartog's method are employed to study a Timoshenko beam with TMDs under a harmonic excitation. The most important mode, usually the fundamental one which dominates the beam vibration, should be taken into account in the design of TMDs for vibrational control, a simplified two-degree-of-freedom system is established and used to predict the design parameters of the tuned mass damper at the optimal condition. Some important and useful formulas are derived, such as the upper bounds of the mass ratio, stiffness, and damping ratio. Finally a series of design charts of the tuned mass damper are presented and they might be very useful in practical design.

2. Equations of motions

A simply supported Timoshenko beam with several tuned mass dampers as shown in Fig. 1 and subjected to a harmonic force is used as an example to set up the equations of motions for the general case.

The symbols shown in this figure are defined as follows: m and J represent the mass and rotary inertia of the mass per unit length of the beam; $k'A$ and I the effective shear area and second moment of area of the beam section; E and G Young's and shear moduli of the beam; m_n, k_n, c_n, and x_n the mass, stiffness, damping, and the position of the nth TMD; M_j and x_j the concentrated mass and its location on the beam; and ℓ the length of the beam, respectively.

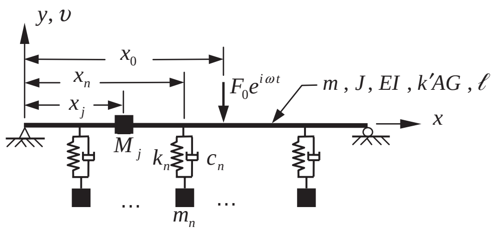

Fig. 1. A Timoshenko beam with TMDs.

The dynamic-stiffness matrix of a Timoshenko-beam element [20] is employed to study the dynamic behaviors of the beam with TMDs. The beam of length ℓ should be divided into several beam elements [23] including the section between two successive TMDs and the section between the support and its nearby TMD. The vertical motion of the mass of each TMD is considered as a degree of freedom of displacement included in analysis. Applying the procedure of the direct stiffness method, the dynamic stiffness matrix of an entire beam structure with all TMDs can be accomplished by superposing the contributions from all of the beam elements and the TMDs affected by each individual nodal displacement. The stiffness equation of the beam with TMDs can be obtained and expressed in the form as used in the standard structural analysis, i.e.,

$$[\mathbf{K}]\{\varDelta\} = \{\mathbf{F}\}, \tag{1}$$

where $[\mathbf{K}]$, $\{\varDelta\}$ and $\{\mathbf{F}\}$ represent the dynamic-stiffness matrix, the nodal displacement vector, and the nodal force vector, respectively.

3. Simplified two-degree-of-freedom system

The most simple way to achieve the vibrational reduction of a beam is to add one or several TMDs to the beam properly. In most cases the fundamental mode or the first important mode of the beam is probably the most important one which should be taken into account in the design of the TMDs. In practice, the ratio of the total mass of TMDs to that of a beam should not be much greater than 10%. Therefore, the change of the vibrational mode shape of the beam due to TMDs could not be significant. For this reason the vibration of the beam with TMDs could be assumed as

$$\begin{aligned}
&\upsilon_i(x,t) \doteq \phi_i(x) y_i(t),\\
&\theta_i(x,t) \doteq \psi_i(x) y_i(t),\\
&\upsilon_{ni}(x,t) = y_{ni}(t), \quad n = 1\text{–}N,
\end{aligned} \tag{2}$$

where $\upsilon_i(x,t)$ and $\theta_i(x,t)$ represent the transverse and rotational displacements of the beam with TMDs; $\phi_i(x)$ and $\psi_i(x)$ the corresponding mode shapes of the beam without TMDs; $y_i(t)$ the generalized co-ordinate (or the amplitude) of the beam with TMDs; $\upsilon_{ni}(t)$ and $y_{ni}(t)$ the vertical displacement and the generalized co-ordinate (or amplitude) of the nth TMD. The subscripts n and i denote the nth TMD and the ith vibrational mode, respectively.

Applying Hamilton's principle, the equations of motion of the beam with TMDs for the ith mode, if a harmonic force $F_0 e^{i\omega t}$ is acting at the position x_0 on the beam, can be obtained and described by the generalized co-ordinates $y_i(t)$ and $y_{ni}(t)$ $(n = 1\text{–}N)$ as following:

$$\begin{bmatrix} m^* & \vdots & \underset{\sim}{0}^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ \underset{\sim}{0} & \vdots & \underset{\sim}{m}_n \end{bmatrix} \begin{Bmatrix} \ddot{y}_i \\ \cdots \\ \ddot{\underset{\sim}{y}}_{n_i} \end{Bmatrix} + \begin{bmatrix} c^* & \vdots & \underset{\sim}{c}_{bt}^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ \underset{\sim}{c}_{bt} & \vdots & \underset{\sim}{c}_n \end{bmatrix} \begin{Bmatrix} \dot{y}_i \\ \cdots \\ \dot{\underset{\sim}{y}}_{ni} \end{Bmatrix} + \begin{bmatrix} k^* & \vdots & \underset{\sim}{k}_{bt}^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ \underset{\sim}{k}_{bt} & \vdots & \underset{\sim}{k}_n \end{bmatrix} \begin{Bmatrix} y_i \\ \cdots \\ \underset{\sim}{y}_{ni} \end{Bmatrix} = \begin{Bmatrix} p^* \\ \cdots \\ \underset{\sim}{0} \end{Bmatrix}_{(N+1)\times 1}, \tag{3}$$

where

$$\begin{aligned} m^* &= \int_0^{\ell} m\phi_i^2(x)\,\mathrm{d}x + \int_0^{\ell} J\psi_i^2(x)\,\mathrm{d}x + \sum_j M_j\phi_i^2(x_j), \\ c^* &= \sum_{n=1}^{N} c_n\phi_i^2(x_n), \\ k^* &= \int_0^{\ell} EI[\phi_i''(x)]^2\,\mathrm{d}x + \int_0^{\ell} k'AG[\phi_i'(x) - \psi_i(x)]^2\,\mathrm{d}x + \sum_{n=1}^{N} k_n\phi_i^2(x_n), \\ p^* &= F_0\phi_i(x_0)e^{i\omega t}, \\ \underset{\sim}{c}_{bt}^{\mathrm{T}} &= -\langle c_1\phi_1(x_1) \quad c_2\phi_2(x_2) \quad \cdots \quad c_N\phi_N(x_N)\rangle, \\ \underset{\sim}{k}_{bt}^{\mathrm{T}} &= -\langle k_1\phi_1(x_1) \quad k_2\phi_2(x_2) \quad \cdots \quad k_N\phi_N(x_N)\rangle, \\ \underset{\sim}{0} &= \textit{zero column matrix}, \\ \underset{\sim}{m}_n,\ \underset{\sim}{c}_n,\ &\textit{and}\ \underset{\sim}{k}_n = \textit{property matrices of TMDs}. \end{aligned} \tag{4}$$

The steady state response of Eq. (3) can be expressed by

$$\begin{Bmatrix} y_i(t) \\ \cdots \\ \underset{\sim}{y}_{ni}(t) \end{Bmatrix}_{(N+1)\times 1} = \begin{Bmatrix} Y \\ \cdots \\ \underset{\sim}{Y}_{ni} \end{Bmatrix} e^{i\omega t}. \tag{5}$$

Since the submatrices $\underline{m_n}$, $\underline{c_n}$, and $\underline{k_n}$ shown in Eq. (3) are all diagonal, every response Y_{ni} $(n = 1\text{–}N)$ of TMDs is related to the response Y only. Therefore Eq. (3) can be reduced to

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} Y \\ Y_m \end{Bmatrix} = \begin{Bmatrix} F_0\phi_i(x_0)/m^* \\ 0 \end{Bmatrix}_{2\times 1}, \tag{6}$$

where Y_m represents the response of a reference TMD located at x_m on the beam.

The $(N+1)$-degree-of-freedom system as described by Eq. (3) can then be simplified to a two-degree-of-freedom system as described by Eq. (6). All the parameters of TMDs could be determined much easily by this simplified two-degree-of-freedom system. The approximation of this simplified two-degree-of-freedom system is based on the assumption that the change of the vibrational mode shape due to TMDs is not significant. In general, this assumed mode shape (i.e., the mode shape of the beam without TMD) would provide a quite good approximation in practice due to the upper limit of the mass ratio of TMDs, usually 15%. Of course, the result can be improved easily by the iterative calculation.

If all TMDs are identical and each has the mass m_t, spring constant k_t, and dashpot damping c_t, K_{11}, K_{12}, K_{21}, and K_{22} are given as follows:

$$\begin{aligned} K_{11} &= \omega_0^2\left[1-\beta^2+\mu\phi_i^2(x_m)f^2+\mathrm{i}2\xi f\beta\mu\phi_i^2(x_m)-\frac{\mu\beta^2(f^2+\mathrm{i}2\xi f\beta)}{f^2-\beta^2+\mathrm{i}2\xi f\beta}\sum_{\substack{n=1\\ n\neq m}}^{N}\phi_i^2(x_n)\right],\\ K_{12} &= -\,\omega_0^2\mu\phi_i(x_m)(f^2+\mathrm{i}2\xi f\beta),\\ K_{21} &= -\,\omega_0^2\phi_i(x_m)(f^2+\mathrm{i}2\xi f\beta),\\ K_{22} &= \omega_0^2(f^2-\beta^2+\mathrm{i}2\xi f\beta), \end{aligned} \tag{7}$$

where

$$\mu=\frac{m_t}{m^*},\quad \xi=\frac{c_t}{2m_t\omega_t},\quad \omega_t=\sqrt{\frac{k_t}{m_t}},$$
$$\omega_0=\sqrt{\frac{k^*}{m^*}},\quad \beta=\frac{\omega}{\omega_0},\quad f=\frac{\omega_t}{\omega_0}. \tag{8}$$

The determinant of the stiffness matrix on the left side of Eq. (6) is given by

$$|K| = \omega_0^4[\beta^4-(1+sf^2)\beta^2+f^2+\mathrm{i}(2\xi f\beta)(1-s\,\beta^2)], \tag{9}$$

where

$$s = 1+\mu\sum_{n=1}^{N}\phi_i^2(x_n). \tag{10}$$

Eq. (6) shows the equations of motions of the simplified two-degree-of-freedom system. The responses Y and Y_m can be obtained easily and given as

$$\begin{aligned} Y &= \frac{F_0\phi_i(x_0)}{k^*}\left(\frac{a+\mathrm{i}b\xi}{c+\mathrm{i}e\xi}\right),\\ Y_m &= \frac{F_0\phi_i(x_0)}{k^*}\phi_i(x_m)\left(\frac{f^2+\mathrm{i}b\xi}{c+\mathrm{i}e\xi}\right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} &a=f^2-\beta^2,\quad b=2f\beta,\\ &c=\beta^4-(1+sf^2)\beta^2+f^2,\quad e=2f\beta(1-s\beta^2). \end{aligned} \tag{12}$$

The response amplitudes of the beam Y and the reference TMD Y_m can be expressed in terms of the static and dynamic terms, i.e.,

$$|Y| = Y_{st}D,\quad |Y_m| = Y_{st}\,D_m, \tag{13}$$

where

$$D = \left(\frac{a^2 + b^2\xi^2}{c^2 + e^2\xi^2}\right)^{1/2},$$

$$D_m = \phi_i(x_m)\left(\frac{f^4 + b^2\xi^2}{c^2 + e^2\xi^2}\right)^{1/2},$$

$$Y_{st} = \frac{F_0}{k^*}\phi_i(x_0). \tag{14}$$

In the foregoing equation D and D_m are defined as the dynamic magnification factor and the displacement transmissibility, respectively.

The force transmissibility at support is defined as the ratio of the dynamic reaction (or the dynamic shear force at the beam end) to the static one.

4. Mass and stiffness of TMD at tuned frequency

No matter the value of the damping ratio ξ of TMD, the D–β curve described by Eq. (14) always pass through two fixed points. These two fixed points can be determined by setting $\xi = 0$ or 1.0, which yields the following condition as

$$\frac{a^2}{c^2} = \frac{b^2}{e^2}. \tag{15}$$

The foregoing condition gives two equations as $ae - bc = 0$ and $ae + bc = 0$, the former one yields $\beta = 0$ which is meaningless; the latter one yields the result as

$$\beta^4 - B\beta^2 + C = 0, \tag{16}$$

where

$$B = \frac{2(1 + sf^2)}{(1 + s)}, \quad C = \frac{2f^2}{(1 + s)}. \tag{17}$$

Only the two positive real roots of Eq. (16) can give the two values of β of the two fixed points on the D–β curve. If these two fixed points are assigned as (β_a, D_a) and (β_b, D_b), substituting Eq. (15) into Eq. (14) and setting $\xi = 0$ gives the condition for $D_a = D_b$ as follows:

$$\left(\frac{b}{e}\right)_{\beta=\beta_a} = \pm\left(\frac{b}{e}\right)_{\beta=\beta_b}. \tag{18}$$

The positive or negative term in the right side of Eq. (18) yields the following results: $\beta_a = \beta_b$ or $\beta_a^2 + \beta_b^2 = 2/s$. The former one is meaningless, and the latter one shows the relationship between the two values of β of the two fixed points for the condition $D_a = D_b$. The sum of two roots of Eq. (16) gives $\beta_a^2 + \beta_b^2 = 2(1 + sf^2)/(1 + s)$. Therefore these two equations of $(\beta_a^2 + \beta_b^2)$ gives the tuned-frequency ratio of each TMD as

$$f_{tuned} = \frac{1}{s} = \frac{1}{1 + \mu r}, \tag{19}$$

where

$$r = \sum_{n=1}^{N} \phi_i^2(x_n). \tag{20}$$

As the effective mass ratio μ decreases and approaches zero, the tuned-frequency ratio of each TMD increases and approaches unity. The tuned-frequency ratio is also dependent of the modal shape ϕ_i and the position x_n where each TMD is attached to the beam.

The relationship between the effective mass ratio μ and the effective stiffness ratio κ, i.e., $\kappa = k_t/k^*$, of each TMD at the tuned condition can be derived from Eq. (19) and given as

$$\mu_{1,2} = \frac{(1/\kappa - 2r) \mp [(1/\kappa - 2r)^2 - 4r^2]^{1/2}}{2r^2}. \tag{21}$$

Only the negative term on the right side of Eq. (21) will give the convergent real value of μ under the following condition:

$$\kappa \leqslant \frac{1}{4r}. \tag{22}$$

The discussions of the application of Eqs. (21) and (22) will be given by a practical example included in this paper.

The effective stiffness ratio κ of each TMD at the tuned condition can be obtained in terms of μ from Eq. (21) and given as

$$\kappa = \frac{\mu}{(1 + \mu r)^2}. \tag{23}$$

Therefore the upper limit of the effective mass ratio μ is given by

$$\mu \leqslant \frac{1}{r}. \tag{24}$$

The co-ordinates of the two fixed points on the D–β curve at the tuned condition can be determined accordingly as

$$\beta_{a,b}^2 = \frac{1}{s} \mp \frac{1}{s}\sqrt{\frac{s-1}{s+1}}, \quad D_a = D_b = \sqrt{\frac{s+1}{s-1}}, \tag{25}$$

where

$$s = 1 + \mu r. \tag{26}$$

5. Optimal damping ratio of TMD

Once the design parameters of each TMD, namely the effective mass ratio μ and the effective stiffness ratio κ at the tuned condition, have been estimated by Eqs. (21) and (23), the dynamic responses or the dynamic-magnification factor D and the displacement transmissibility D_m can be all calculated by Eq. (1) or (14) for any value of the damping ratio ξ of each TMD. All the D–β curves for the different damping ratio ξ pass the two fixed points of equal D. If the peaks of the D–β curve locate at and coincide with these two fixed points for a certain value of ξ, it will result in the minimum response of the beam, and this damping ratio is called the optimal damping ratio of each TMD [1]. In general cases both the two peaks of the D–β curve could not coincide with the two fixed points simultaneously for any value of ξ, but they might be very close to each other for a certain value of ξ. Eq. (14) can be rewritten as

$$\xi^2 = \frac{c^2D^2 - a^2}{b^2 - e^2D^2}, \tag{27}$$

where a, b, c, and e are all functions of β.

Replacing β^2 by $(\beta^2 + \varepsilon)$, where ε is a very small number, and omitting all the high-order terms of ε, finally Eq. (27) becomes

$$\xi^2 = \frac{(s-1)[3-(1+2s)s\beta^2]}{4s[2-(1+s)s\beta^2]}. \tag{28}$$

The two values of β^2 assigned as β_a^2 and β_b^2 at the two fixed points of equal D (or at the tuned condition) is given by Eq. (25). Therefore the damping ratio for the maximum of D located at each one of the two fixed points are given as

$$\xi_{a,b}^2 = \frac{(s-1)\left[3 \mp \sqrt{(s-1)/(s+1)}\right]}{8s}. \tag{29}$$

According to Hartog's method [1], the average of ξ_a^2 and ξ_b^2 could give the optimal damping ratio of TMD as

$$\xi_0 = [3(s-1)/8s]^{1/2}. \tag{30}$$

6. Example and discussion

A simply supported rectangular steel beam has the following properties: beam length $\ell = 100$ cm, beam width $b = 3$ cm, beam depth $d = 2$ cm, beam density $\rho = 7800\ \mathrm{kg/m^3}$, Young's modulus $E = 2 \times 10^{11}\ \mathrm{N/m^2}$, the Poisson ratio $\nu = 0.3$, and the shape factor for shear $k' = 0.87$. The fundamental mode of the beam is taken into account in the design of TMDs for the vibrational control by adding one, three or five identical TMDs to the beam as shown in Fig. 2 under investigation. The total mass of TMDs is constant for each case and equals 10% of the beam mass. The tuned stiffness and optimal damping ratio of each TMD are determined by Eqs. (23) and (30), and the results are given in Table 1. The natural frequencies (ω_i's), modal damping ratios (η_i's), and mode shapes of the six lowest modes of the beam with or without TMD are all shown in Fig. 2. The natural frequencies shown inside the brackets are the results of the simplified two-degree-of-freedom system using the mode shape of the beam without TMDs, and they are very close to the exact solutions. It can be obviously found in Fig. 2 that the fundamental mode of the beam without TMD will be split into two independent modes, when TMDs are designed and attached to this beam. All the high-mode natural frequencies are very close to each other of the beam with or without TMD. It can also be seen that all the mode shapes of these three cases look very similar each other and their differences are not significant due to TMDs. The percentages of the values of the modal damping ratios (η_i's) are also given in Fig. 2. The Timoshenko beam is assumed undamped, the damping is only coming from TMDs. The modal

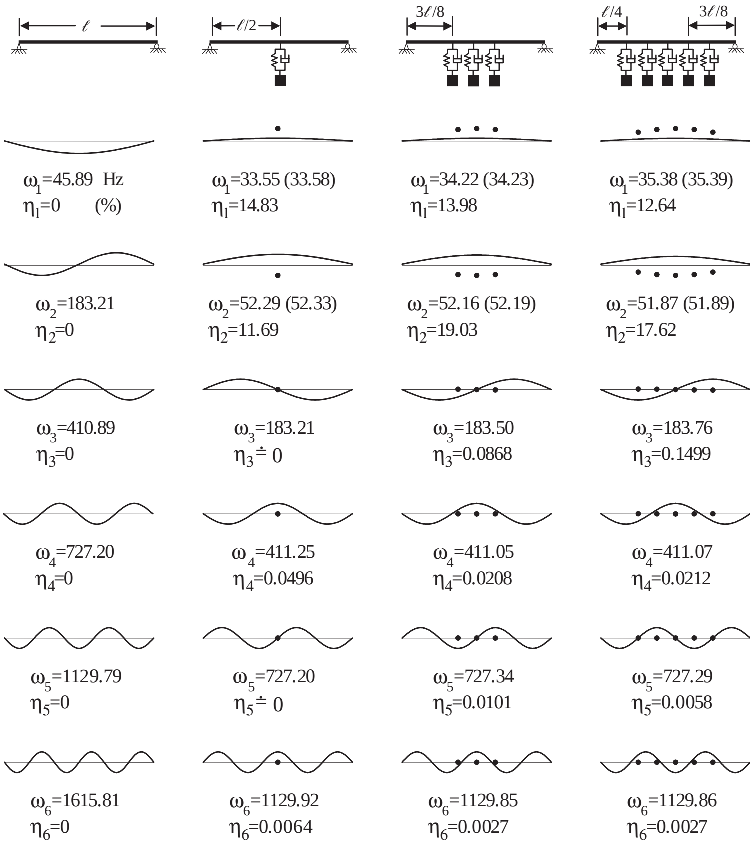

Fig. 2. Mode shapes.

Table 1
Three cases of a simply supported beam with TMDs

Case	$\mu_t = \dfrac{m_t}{m\ell}$	$\mu = \dfrac{m_t}{m^*}$	$\kappa = \dfrac{k_t}{k^*}$	k_t (N/m)	ξ_o	$\mu_{t,max}$	$k_{t,max}$	ξ_{max}
TMD-1	0.100	0.200	0.1389	27058.08	0.250	0.5000	48704.55	0.4329
TMD-3	0.033	0.067	0.0478	9320.20	0.239	0.1847	17991.36	0.4329
TMD-5	0.020	0.040	0.0303	5910.04	0.220	0.1349	13138.15	0.4329

damping ratios of the Timoshenko beam with TMDs are about 12–20% for the first two vibrational modes, and they are very small and less than about 0.1% for the higher modes. It means that TMD is very effective for the fundamental or the first important mode of the beam which is desired to be controlled, and it is not effective for the higher modes.

If a unit harmonic force is applied at the center of the beam which is the most important case in general. The dynamic magnification factor D at the center of the beam, the displacement transmissibility D_m of the middle TMD, and the force transmissibility T_f at the left support are shown in Figs. 3, 4 and 5, respectively. It can be seen that the single-TMD case (or the case of

TMD-1) is not only the most effective to control the beam vibration but also beneficial to the TMDs vibration and the support reaction. The higher-frequency (or higher-mode) performance of TMDs are also shown in Figs. 3–5. The results of D and T_f seem to be too high, this is due to the facts that the beam is presumed undamped and TMDs are designed for the first mode; therefore TMDs are much less effective for the higher modes. Actually, the beam-own damping will play an important role for higher modes and suppress the beam vibration within a certain limit. The results of D, D_m, and T_f in these figures can also reflect an important fact that the performance of the case of TMD-1 is the best for both the low-mode as well as the high-mode vibrations. The small beam-own damping could not influence the results of the low-mode vibration significantly, but it will dominate the high-mode vibration. The structural analysis of a Timoshenko beam with TMDs can include the effect of the beam-own damping and the responses can be calculated numerically, however the explicit formulas presented in this paper would still provide a good initial approximation of the design of TMDs.

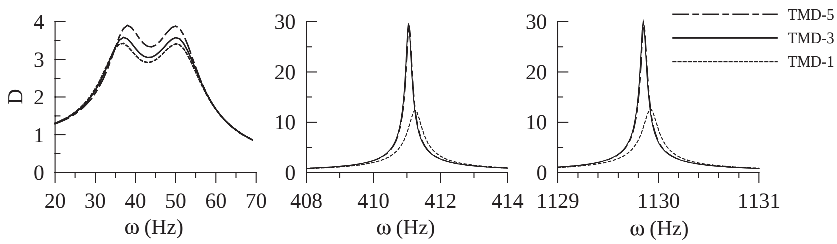

Fig. 3. Dynamic magnification factor at the midspan of the beam.

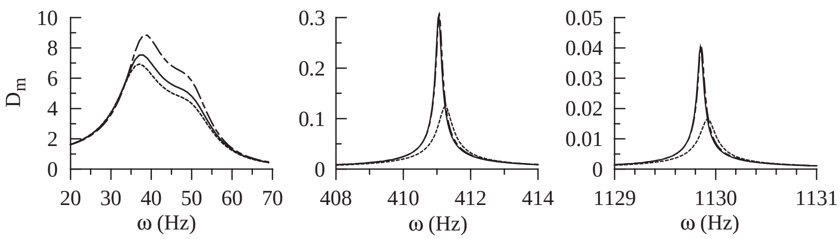

Fig. 4. Displacement transmissibility of the middle TMD.

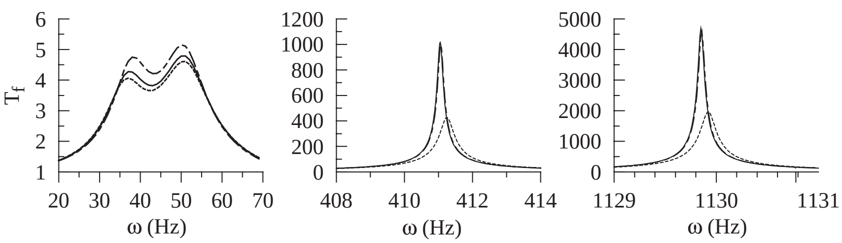

Fig. 5. Force transmissibility at each support of the beam.

The relationship between the effective mass and stiffness ratios, μ and κ, of TMDs is given by Eq. (21). μ_1 and μ_2 are calculated by use of Eq. (21) and the results are shown in Fig. 6 for the case of a single TMD at the center of the beam (namely, the case of TMD-1). Both μ_1 and μ_2 are real, if the stiffness of TMD is less than a certain limit, i.e., $k_t \leqslant k_{t,\,max} = 48704.55$ N/m calculated by Eq. (22). Only μ_1 is convergent and useful for the practical design of TMD. Both μ_1 and μ_2 are complex if $k_t < k_{t,\,max}$, it means that the stiffness of TMD is too strong to absorb any vibrational energy from the beam, and both the beam and TMD will move together for this situation. The μ_1–k curves for all cases assigned as TMD-1, TMD-3 and TMD-5 are all shown in Fig. 7. The maximum values of the mass, stiffness, and damping ratio of each TMD are calculated by Eqs. (24), (22) and (30), and the results are also given in Table 1.

If we consider the mass ratio (μ_t), stiffness (k_t), and damping ratio (ξ) of each TMD as the design parameters, and D, D_m, and T_f as the control indexes for a specific loading condition such as a unit harmonic load acting at the midspan of the beam for an example, the design curves of

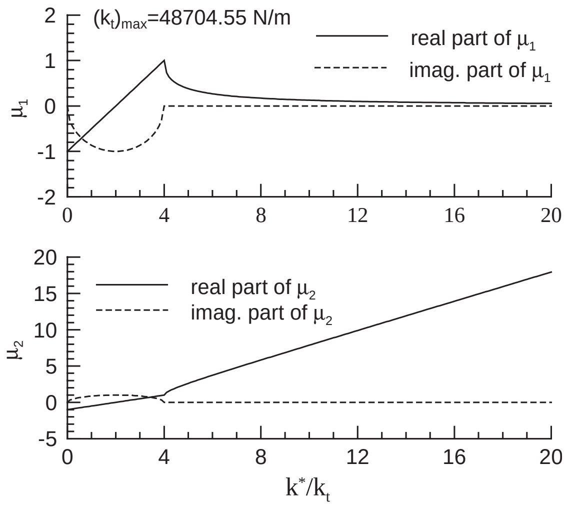

Fig. 6. Relationship between the effective mass ratio μ and effective stiffness ratio κ (TMD-1).

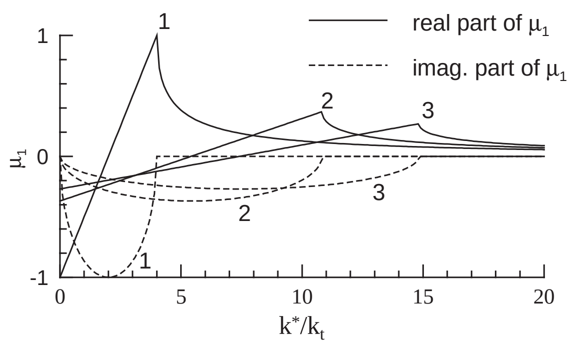

Fig. 7. Relationship between the effective mass ratio μ and effective stiffness ratio κ: (1) TMD-1, (2) TMD-3, and (3) TMD-5.

TMD for the three cases of TMD-1, TMD-2, and TMD-3 can be established and shown in Figs. 8–11. These design curves can be very convenient and useful in practice. It can be seen that the dynamic magnification factor D is almost inversely proportional to the design parameters μ_t, k_t and, ξ. The increasing ratio of μ_t becomes very high if μ_t is greater than a certain limit (say about 15%) and beyond this limit the decreasing rate of D becomes very small. This means that it is not economical to design a heavy TMD or several heavy TMDs (say $\mu_t > 15\%$). The design will fail if μ_t or k_t is greater than its maximum value (or called the design limit). This is the reason that both the beam and TMDs will move in the same direction due to the hard spring of high value of k_t tying the beam and TMDs together while vibration, and thus the energy absorbed by TMDs is very limited. The relationship between ξ and k_t is almost linear in the most practical design range, and the relationship between μ_t and k_t is also almost linear as $\mu_t \leqslant 10\%$ (see Figs. 8–11). It is very interesting to note from Figs. 8–10 that the reaction force at beam support has a minimum at a certain condition. In order to avoid much of the vibrational energy of the beam transmitting to the beam support and keep the environment in quiet condition, the curve of T_f as shown in these figures can provide an important design guide for this purpose. The minimum of the left-support reaction (T_f) lies near to the center of the design range of k_t as shown in Figs. 8–10, the

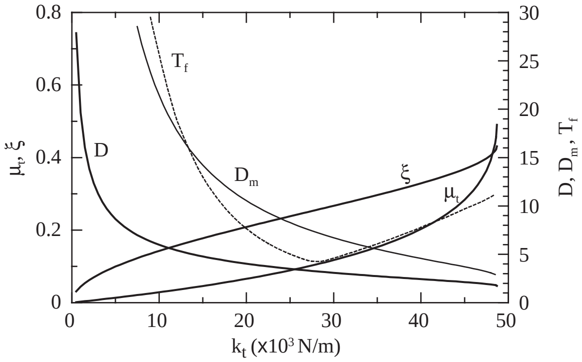

Fig. 8. Design curve (TMD-1, force at $\chi_0 = \ell/2$).

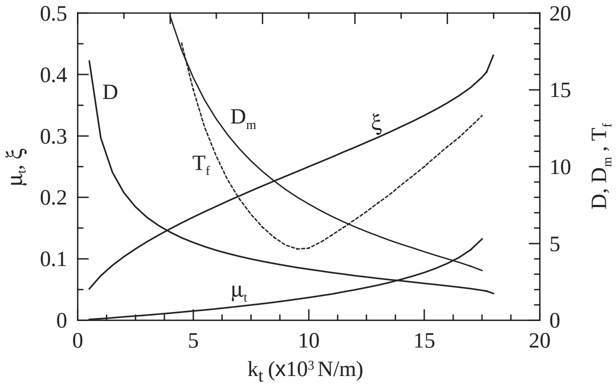

Fig. 9. Design curve (TMD-3, force at $\chi_0 = \ell/2$).

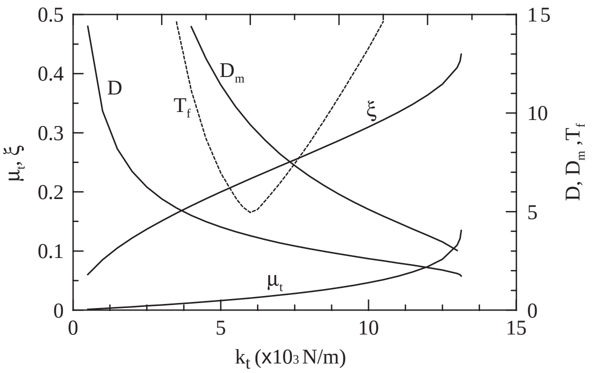

Fig. 10. Design curve (TMD-5, force at $\chi_0 = \ell/2$).

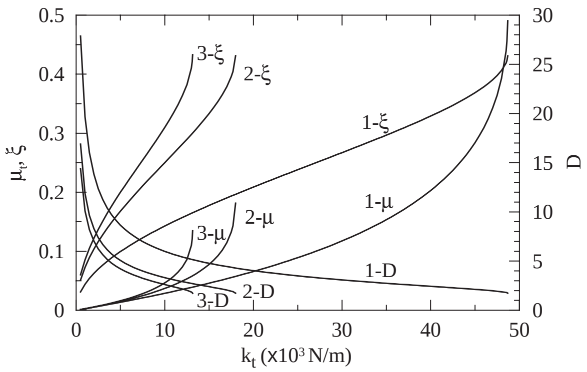

Fig. 11. Design curves: (1) TMD-1, (2) TMD-3, and (3) TMD-5, force at $\chi_0 = \ell/2$.

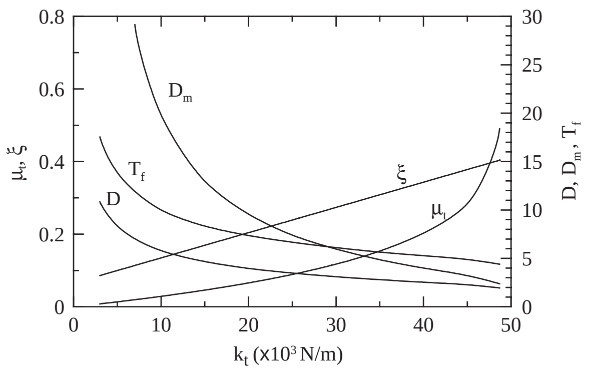

Fig. 12. Design curve (TMD-1, moment at $\chi_0 = \ell/4$).

corresponding values of μ_t, D, ξ, *and* D_m seem to be reasonable in practical design, especially for the case of TMD-1 which is the most effective compared with the other two as mentioned previously.

If a harmonic moment $M_o e^{iwt}$ is applied at the position x_0 on the beam, the generalized force given by Eq. (4) should be replaced by $P^* = M_o \psi_i(x_i) e^{iwt}$. All formulas presented in this paper are still valid except the terms P_o and $\phi_i(x_0)$ should be replaced by M_o and $\psi_i(x_i)$, respectively. The design curves for the case TMD-1 are shown in Fig. 12, if a unit harmonic moment is acting at the quarter of the beam i.e., $M_o = 1$ and $x_o = \ell/4$. All the corresponding values of μ_t, ξ, D, D_m, *and* T_f also seem to be reasonable in practice, when the value of k_t is about the half of its maximum value $k_{t,max}$.

7. Conclusions

Some important conclusions would be drawn from this study and given as follows:

(1) The dynamic stiffness matrix of a Timoshenko beam and the approximate model of the simplified two-degree-of-freedom system would be successfully employed for the structural analysis and design of a Timoshenko beam with TMDs in practice for the whole frequency range including the low-frequency (or low-mode) as well as the high-frequency (or high-mode) vibrations.
(2) The approximate mode shape of the Timoshenko beam without TMD can be employed to establish the simplified two-degree-of-freedom system to predict the design parameters of TMDs, if the total mass ratio of TMDs is less than a certain limit (say 15%). Of course, the results can be improved by the iterative calculation.
(3) The tuned frequency and the optimal damping ratio of TMDs, and the upper bounds of the TMDs properties (i.e., the mass, stiffness, and damping) can be predicted either by Hartog's method or by Jacquot and Hoppe's method. The former one is based on the minimization of the amplitude of the steady state response under the harmonic excitation, whereas the latter one is based on the minimization of the mean-square response under the white-noise random excitation.
(4) The natural frequencies and mode shapes of a Timoshenko beam are the basic important informations for the design of TMDs to control the beam vibration. Whether a single or multiple TMDs should be designed to control the beam vibration, it should depend on the mode shape of the beam which is desired to be controlled. Of course, the position of the harmonic exciting force or moment and the control point on the beam should also be taken into account.
(5) There is an upper bound of the mass ratio of TMDs. The design will fail if the mass ratio of TMDs is beyond this limit. It means that the required spring of TMDs for this case is too strong to absorb the energy from the beam vibration. Both the beam and TMDs will move together in the same direction due to the hard spring of high stiffness tying them together while vibration.
(6) It is not economic to design a heavy TMD system, say total $\mu_t \geqslant 15\%$. The decreasing rate of D of the beam vibration becomes small, but on the contrary the increasing rate of μ_t becomes tremendously large as μ_t beyond this limit.
(7) A good design can provide both the vibrations of the beam and TMDs as small as possible, not only for the low frequency, but also for the high frequency.

(8) The design ranges of the TMDs properties, i.e., the mass, and stiffness, become smaller and less flexible, if more TMDs are used (also see Fig. 11).

(9) The design curves presented in this paper would provide the important informations for the design of the tuned-mass dampers attached to a Timoshenko beam for vibrational control. All the corresponding values of $\mu_t,\ \xi,\ D,\ D_m,$ *and* T_f seem to be reasonable and acceptable, if the value of k_t is about the half of its maximum value $k_{t,\,max}$. This might be a design guide in practice, however the final design of TMDs should be made by the designer's experience and judgement.

(10) There are two damping components of a Timoshenko beam system with TMDs, one is coming from TMDs and other from the beam structure. The latter can be included in structural analysis and the response can be calculated numerically, but the explicit formulas might be difficult to be derived. However the formulas presented in this paper can still provide a good approximation at first before the design of TMDs, if the damping of the beam structure should be included.

(11) If all TMDs are not identical, Eq. (6) is still valid for this case. The formulas presented in this paper should be modified and it is considered as an extension or the future study of this paper.

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